ON THE GRADIENT OF SOLID HARMONIC POLYNOMIALS*

BY

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To Leopold Fejér
on his 60th birthday, February 9, 1940.

Introduction

In 1912 S. Bernstein proved a theorem on trigonometric polynomials which has been considered by various authors during subsequent years.† For our later purposes it is convenient to formulate this theorem in terms of harmonic polynomials as follows:

I. *Let* $u(x, y)$ *be a real harmonic polynomial of the nth degree which satisfies the inequality* $|u(x, y)| \leqq 1$ *in the unit circle* $x^2+y^2 \leqq 1$. *If* l *denotes the positive tangential direction at an arbitrary point* (x, y) *of the unit circle* $x^2+y^2=1$, *we have*

$$\left|\frac{\partial u}{\partial l}\right| \leqq n, \qquad x^2 + y^2 = 1. \tag{1}$$

The equality sign holds only if $u(x, y) = \cos n(\phi - \phi_0)$ *for* $x^2+y^2=1$, *where* $(1, \phi)$ *are the polar coordinates of the point with the cartesian coordinates* (x, y) *on the unit circle, and* ϕ_0 *is arbitrary real.*

Ten years ago I proved the following theorem, which is more informative than I:‡

II. *Let* $u(x, y)$ *be a real harmonic polynomial of the nth degree which satisfies the inequality* $|u(x, y)| \leqq 1$ *in the unit circle* $x^2+y^2 \leqq 1$. *Then*

$$|\operatorname{grad} u| = (u_x^2 + u_y^2)^{1/2} \leqq n, \qquad x^2 + y^2 \leqq 1. \tag{2}$$

The equality sign holds for the same polynomials as in I; *it holds only for* $x^2+y^2=1$ *if* $n>1$.

In the present investigation I am dealing with the three-dimensional analogue of the last theorem. The following result holds:

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† See the literature quoted in the introduction of the paper referred to in the next footnote.

‡ G. Szegö, *Über einen Satz des Herrn Bernstein*, Schriften der Königsberger Gelehrten Gesellschaft, 1928, pp. 59–70. See also S. Bernstein, *Sur un théorème de M. Szegö*, Prace Matematyczno-Fizyczne, vol. 44 (1937), pp. 9–14.

III. Let $u(x, y, z)$ *be a real harmonic polynomial of the nth degree which satisfies the inequality* $|u(x, y, z)| \leqq 1$ *in the unit sphere* $x^2+y^2+z^2 \leqq 1$. *We assume* $n \geqq 4$. *Then*

$$(3)\qquad \begin{aligned} |\operatorname{grad} u| &= (u_x^2 + u_y^2 + u_z^2)^{1/2} \\ &\leqq 2n\left(\frac{1}{1} - \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{2n-1}\right) + \begin{cases} 0 & \textit{if } n \textit{ even}, \\ -1 & \textit{if } n \textit{ odd}, \end{cases} \\ & \qquad x^2 + y^2 + z^2 \leqq 1. \end{aligned}$$

The equality sign holds only for the polynomial $u(x, y, z)$ *whose boundary values on the unit sphere* $x^2+y^2+z^2=1$ *are of the form* $\pm\cos n\gamma$; *here* γ *is the spherical distance of the variable point* (x, y, z) *on the unit sphere from a fixed point* P_0 *of this sphere.*

For these polynomials the equality sign holds only at P_0 *and at the point diametral to* P_0.

The bound ρ_n in the inequality (3) is the precise one. We have $\rho_n \cong n\pi/2$ as $n \to \infty$.

The proof follows an argument very much similar to that used in the proof of my former Theorem II. The underlying idea is due essentially to M. and F. Riesz; these authors dealt, of course, only with the inequality (1) of S. Bernstein. In the present case the chief difficulty is to prove the positivity of the trigonometric polynomial

$$(4)\qquad \begin{aligned} M_n(\theta) = 1 + 2\sum_{\nu=1}^{n-1} \bigg\{ &\frac{\rho_{n-\nu}}{\rho_n} \cos n\theta \cos (n-\nu)\theta \\ &+ [(n-\nu)/n] \sin n\theta \sin (n-\nu)\theta \bigg\} \end{aligned}$$

for all real values of θ.

Unfortunately, my proof for (3) fails if $n=2$ or $n=3$. In these cases the less precise inequality

$$(3')\qquad |\operatorname{grad} u| \leqq 2^{1/2} n$$

can be obtained. We notice that $2^{1/2}\cdot 2 = 2.82 \cdots$ whereas $\rho_2 = 8/3 = 2.66 \cdots$, and $2^{1/2}\cdot 3 = 4.23 \cdots$ whereas $\rho_3 = 21/5 = 4.20$.

First part of the proof

1. It is sufficient to prove (3) for $x=y=0$, $z=1$. For the other points of the unit sphere the statement follows by a simple rotation, for the interior points by considering $u(kx, ky, kz)$, $0 \leqq k < 1$.

If we introduce polar coordinates r, θ, ϕ (in the usual notation), we have for $u(x, y, z) = U(r, \theta, \phi)$

$$|\operatorname{grad} u| = \{U_r^2 + (r^{-1}U_\theta)^2 + ((r \sin\theta)^{-1}U_\phi)^2\}^{1/2}, \tag{5}$$

so that our inequality (3) is equivalent to the following statement: Let α, β be arbitrary real constants, and

$$K = \cos\alpha \cdot U_r + \sin\alpha\cos\beta \cdot r^{-1}U_\theta + \sin\alpha\sin\beta(r\sin\theta)^{-1}U_\phi; \tag{6}$$

then

$$|K| \leqq \rho_n. \tag{7}$$

2. Representing $u(x, y, z) = U(r, \theta, \phi)$ in the form

$$U(r, \theta, \phi) = A_0(r, \theta) + \sum_{\nu=1}^{n} (A_\nu(r, \theta)\cos\nu\phi + B_\nu(r, \theta)\sin\nu\phi), \tag{8}$$

we have

$$\begin{aligned} A_\nu(r, \theta) &= \sin^\nu\theta \sum_{\mu=\nu}^{n} a_{\mu\nu} P_\mu^{(\nu)}(\cos\theta) r^\mu, \\ B_\nu(r, \theta) &= \sin^\nu\theta \sum_{\mu=\nu}^{n} b_{\mu\nu} P_\mu^{(\nu)}(\cos\theta) r^\mu, \end{aligned} \tag{9}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are real constants; $P_\mu(\cos\theta)$ is Legendre's polynomial in the usual notation.

For $x = y = 0$, $z = 1$, or $r = 1$, $\theta = 0$, we find

$$\begin{aligned} U_r &= \frac{\partial A_0}{\partial r} = \sum_{\mu=1}^{n} \mu a_{\mu 0}, \\ r^{-1}U_\theta &= \frac{\partial A_1}{\partial\theta}\cos\phi + \frac{\partial B_1}{\partial\theta}\sin\phi \\ &= \sum_{\mu=1}^{n} P_\mu'(1)(a_{\mu 1}\cos\phi + b_{\mu 1}\sin\phi), \\ \lim_{\theta\to 0}((r\sin\theta)^{-1}U_\phi) &= -\frac{\partial A_1}{\partial\theta}\sin\phi + \frac{\partial B_1}{\partial\theta}\cos\phi \\ &= \sum_{\mu=1}^{n} P_\mu'(1)(-a_{\mu 1}\sin\phi + b_{\mu 1}\cos\phi), \end{aligned}$$

so that

$$\begin{aligned} K = \cos\alpha \sum_{\mu=1}^{n} \mu a_{\mu 0} &+ \sin\alpha\cos(\phi+\beta)\sum_{\mu=1}^{n} P_\mu'(1) a_{\mu 1} \\ &+ \sin\alpha\sin(\phi+\beta)\sum_{\mu=1}^{n} P_\mu'(1) b_{\mu 1}. \end{aligned} \tag{10}$$

3. According to the orthogonality property of the surface harmonics, we have

$$\int_0^\pi \int_{-\pi}^{+\pi} U(1, \theta', \phi') P_\mu(\cos \theta') \sin \theta' d\theta' d\phi' = \frac{4\pi}{2\mu + 1} a_{\mu 0},$$

$$\int_0^\pi \int_{-\pi}^{+\pi} U(1, \theta', \phi') \sin \theta' P_\mu' (\cos \theta') \begin{matrix} \cos \phi' \\ \sin \phi' \end{matrix} \sin \theta' d\theta' d\phi' = \frac{4\pi}{2\mu + 1} \cdot \begin{cases} P_\mu'(1) a_{\mu 1}, \\ P_\mu'(1) b_{\mu 1}, \end{cases}$$

so that

$$\begin{aligned} K = \frac{1}{4\pi} \int_0^\pi \int_{-\pi}^{+\pi} U(1, \theta', \phi') \Big\{ &\cos \alpha \sum_{\mu=1}^{n} \mu(2\mu + 1) P_\mu(\cos \theta') \\ &+ \sin \alpha \cos (\phi - \phi' + \beta) \sum_{\mu=1}^{n} (2\mu + 1) \sin \theta' P_\mu' (\cos \theta') \Big\} \sin \theta' d\theta' d\phi'. \end{aligned} \tag{11}$$

In these integrals, $0 \leqq \theta' \leqq \pi$, $-\pi \leqq \phi' \leqq +\pi$.

4. Now, we introduce the Fourier expansions

$$\begin{aligned} &\frac{1}{4} \pi \sin \theta \sum_{\mu=1}^{n} \mu(2\mu + 1) P_\mu(\cos \theta) \\ &\qquad = \rho_1 \cos \theta + \rho_2 \cos 2\theta + \cdots + \rho_n \cos n\theta + \cdots, \\ &\frac{1}{8} \pi \sin^2 \theta \sum_{\mu=1}^{n} (2\mu + 1) P_\mu' (\cos \theta) \\ &\qquad = \sigma_1 \sin \theta + \sigma_2 \sin 2\theta + \cdots + \sigma_n \sin n\theta + \cdots. \end{aligned} \tag{12}$$

Since

$$\int_0^\pi \sin \theta P_\mu(\cos \theta) \cos \nu\theta d\theta = \int_0^\pi \sin^2 \theta P_\mu' (\cos \theta) \sin \nu\theta d\theta = 0, \qquad \mu > \nu,$$

the coefficients ρ_ν, σ_ν, $1 \leqq \nu \leqq n$, depend only on ν; that is to say, for a fixed value of ν, they are independent of n. Consequently, we find from (11):

$$\begin{aligned} K = \frac{1}{2\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} U(1, \theta', \phi') \Big\{ &\cos \alpha \sum_{\nu=1}^{n} \rho_\nu \cos \nu\theta' \\ &+ 2 \sin \alpha \cos (\phi - \phi' + \beta) \sum_{\nu=1}^{n} \sigma_\nu \sin \nu\theta' \Big\} d\theta' d\phi'. \end{aligned} \tag{13}$$

Here both θ', ϕ' run from $-\pi$ to $+\pi$; we use the fact that

$$\int_{-\pi}^{+\pi} U(1, \theta', \phi') d\phi'$$

is a cosine polynomial of the nth degree, and also that both integrals

$$\int_{-\pi}^{+\pi} U(1, \theta', \phi') \begin{matrix} \cos \phi' \\ \sin \phi' \end{matrix} d\phi'$$

represent sine polynomials of the nth degree.

Second part of the proof

The basic idea of this part is similar to that used in the proof of II (see the Introduction).

1. We write (13) in the form

$$K = \frac{1}{2\pi^2} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} U(1, \theta', \phi') \{ \cos \alpha \cdot \rho(\theta') + 2 \sin \alpha \cos (\phi - \phi' + \beta) \cdot \sigma(\theta') \} d\theta' d\phi', \tag{13'}$$

where

$$\rho(\theta') = \sum_{\nu=1}^{n} \rho_\nu \cos \nu\theta', \qquad \sigma(\theta') = \sum_{\nu=1}^{n} \sigma_\nu \sin \nu\theta'. \tag{14}$$

Let us put

$$\rho_n \cos \alpha \cos n\theta' + 2\sigma_n \sin \alpha \cos (\phi - \phi' + \beta) \sin n\theta' = R \cos (n\theta' - \delta), \tag{15}$$

where $R > 0$, δ real, and R, δ are independent of θ'; obviously, R and δ depend on $\alpha, \beta, \phi, \phi', n$. We have

$$R = \{ \rho_n^2 \cos^2 \alpha + 4\sigma_n^2 \sin^2 \alpha \cos^2 (\phi - \phi' + \beta) \}^{1/2}. \tag{16}$$

Furthermore, for $1 \leqq \nu \leqq n-1$,

$$\begin{aligned} \rho_{n-\nu} \cos \alpha \cos (n - \nu)\theta' &+ 2\sigma_{n-\nu} \sin \alpha \cos (\phi - \phi' + \beta) \sin (n - \nu)\theta' \\ &= (\rho_{n-\nu}/\rho_n) R \cos \delta \cos \{ (n\theta' - \delta) - (\nu\theta' - \delta) \} \\ &\quad + (\sigma_{n-\nu}/\sigma_n) R \sin \delta \sin \{ (n\theta' - \delta) - (\nu\theta' - \delta) \}. \end{aligned}$$

Adding to this expression

$$(\rho_{n-\nu}/\rho_n) R \cos \delta \cos \{ (n\theta' - \delta) + (\nu\theta' - \delta) \} - (\sigma_{n-\nu}/\sigma_n) R \sin \delta \sin \{ (n\theta' - \delta) + (\nu\theta' - \delta) \},$$

we obtain

$$2R \cos (n\theta' - \delta) \{ (\rho_{n-\nu}/\rho_n) \cos \delta \cos (\nu\theta' - \delta) - (\sigma_{n-\nu}/\sigma_n) \sin \delta \sin (\nu\theta' - \delta) \}.$$

Consequently, since $U(1, \theta', \phi')$ is a trigonometric polynomial of the nth degree in θ',

$$K = \frac{1}{2\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} U(1, \theta', \phi')\cdot R \cos(n\theta' - \delta) \tag{17}$$
$$\cdot\left\{1 + 2\sum_{\nu=1}^{n-1} \frac{\rho_{n-\nu}}{\rho_n}\cos\delta\cos(\nu\theta' - \delta) - \frac{\sigma_{n-\nu}}{\sigma_n}\sin\delta\sin(\nu\theta' - \delta)\right\} d\theta' d\phi'.$$

2. We now replace $\cos(n\theta' - \delta)$ by

$$\frac{1}{4r}\left\{\frac{1 - r^2}{1 - 2r\cos(n\theta' - \delta) + r^2} - \frac{1 - r^2}{1 + 2r\cos(n\theta' - \delta) + r^2}\right\}, \qquad 0 < r < 1,$$

and obtain in the usual manner*

$$\begin{aligned}
K = {} & \lim_{r\to 1-0}\frac{1}{8\pi^2 r}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} U(1, \theta', \phi') \\
& \cdot R\left\{\frac{1 - r^2}{1 - 2r\cos(n\theta' - \delta) + r^2} - \frac{1 - r^2}{1 + 2r\cos(n\theta' - \delta) + r^2}\right\} \\
& \cdot\left\{1 + 2\sum_{\nu=1}^{n-1}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos\delta\cos(\nu\theta' - \delta)\right.\right. \\
& \left.\left. - \frac{\sigma_{n-\nu}}{\sigma_n}\sin\delta\sin(\nu\theta' - \delta)\right)\right\} d\theta' d\phi' \\
= {} & \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}' U(1, \theta', \phi') R\left\{1 + 2\sum_{\nu=1}^{n-1}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos\delta\cos(\nu\theta' - \delta)\right.\right. \\
& \left.\left. - \frac{\sigma_{n-\nu}}{\sigma_n}\sin\delta\sin(\nu\theta' - \delta)\right)\right\} d\phi' \\
& - \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}'' U(1, \theta', \phi') R\left\{1 + 2\sum_{\nu=1}^{n-1}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos\delta\cos(\nu\theta' - \delta)\right.\right. \\
& \left.\left. - \frac{\sigma_{n-\nu}}{\sigma_n}\sin\delta\sin(\nu\theta' - \delta)\right)\right\} d\phi'.
\end{aligned} \tag{18}$$

The summations $\sum'$, $\sum''$ are extended over the modulo 2π incongruent values of θ' for which $n\theta' - \delta \equiv 0$ and π, respectively. (These particular values of θ' depend, of course, on $\alpha, \beta, \phi, \phi', n$.) For these values

$$\begin{aligned}
& 1 + 2\sum_{\nu=1}^{n-1}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos\delta\cos(\nu\theta' - \delta) - \frac{\sigma_{n-\nu}}{\sigma_n}\sin\delta\sin(\nu\theta' - \delta)\right) \\
& = 1 + 2\sum_{\nu=1}^{n-1}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos n\theta'\cos(n - \nu)\theta' + \frac{\sigma_{n-\nu}}{\sigma_n}\sin n\theta'\sin(n - \nu)\theta'\right) = M_n(\theta').
\end{aligned}$$

* Szegö, loc. cit., p. 64.

Since $\sigma_n = n$ (see (29)), this is the same trigonometric polynomial of order $2n-1$ as defined in (4). It depends only on n.

Consequently,

$$K = \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}' U(1, \theta', \phi') R M_n(\theta') d\phi' \tag{19}$$
$$- \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}'' U(1, \theta', \phi') R M_n(\theta') d\phi'.$$

Here and in the next section, $\sum{}'$ and $\sum{}''$ have the meaning given above.

3. In the last part of the proof we are going to show that $M_n(\theta') > 0$ for all real values of θ'. Anticipating this result, we obtain from (19)

$$|K| \leqq \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}' R M_n(\theta') d\phi' + \frac{1}{4\pi n}\int_{-\pi}^{+\pi} \sum{}'' R M_n(\theta') d\phi'. \tag{20}$$

But we easily see that the trigonometric polynomial $M_n(\theta')$ does not contain any term with $\cos n\theta'$ and $\sin n\theta'$, and also that its absolute term is 1. Hence

$$\frac{1}{n}\sum{}' M_n(\theta') = \frac{1}{n}\sum{}'' M_n(\theta') = 1, \tag{21}$$

so that on account of (16),

$$|K| \leqq \frac{1}{4\pi}\int_{-\pi}^{+\pi} R d\phi' + \frac{1}{4\pi}\int_{-\pi}^{+\pi} R d\phi' \tag{22}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} (\rho_n^2 \cos^2\alpha + 4\sigma_n^2 \sin^2\alpha \cos^2\phi')^{1/2} d\phi'.$$

By means of Schwarz's inequality

$$K^2 \leqq \frac{1}{2\pi}\int_{-\pi}^{+\pi} (\rho_n^2 \cos^2\alpha + 4\sigma_n^2 \sin^2\alpha \cos^2\phi') d\phi' = \rho_n^2 \cos^2\alpha + 2\sigma_n^2 \sin^2\alpha \leqq \rho_n^2,$$

provided that $\rho_n^2 \geqq 2\sigma_n^2$. This is the case for $n \geqq 4$ (see (30)). For these values of n, the inequality $|K| \leqq \rho_n$ is proved.

4. The preceding argument, combined with (30), shows that $|K| < \rho_n$ holds provided that $\sin^2\alpha > 0$. Now let $\sin^2\alpha = 0$, $\cos\alpha = 1$; then $\delta = 0$, $R = \rho_n$. In this case (19) reads as follows:

$$K = \frac{\rho_n}{2n}\sum{}' U(\theta') M_n(\theta') - \frac{\rho_n}{2n}\sum{}'' U(\theta') M_n(\theta').$$

Here $U(\theta') = (2\pi)^{-1}\int_{-\pi}^{+\pi} U(1, \theta', \phi') d\phi'$, and the sums $\sum{}'$, $\sum{}''$ are extended over the modulo 2π incongruent solutions of the equations $n\theta' \equiv 0$ and π, respectively. Taking (21) into consideration, we see that $K = \rho_n$ holds if and only if $U(\theta') = \cos n\theta'$ for the special values of θ' mentioned. Since the cosine polynomial $U(\theta') - \cos n\theta'$ vanishes for the values $\theta' = 0, \pi/n, 2\pi/n, \cdots, \pi$ (for which $\cos\theta'$ assumes $n+1$ distinct values), it must vanish identically, that is, $U(\theta) \equiv \cos n\theta$.

Similarly $K = -\rho_n$ holds if and only if $U(\theta) = -\cos n\theta$.

Considering the case $U(\theta) = \cos n\theta$, we write again $\theta = \theta' = k\pi/n$, k an integer. This furnishes

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} U(1, \theta', \phi')d\phi' = (-1)^k.$$

Since $|U(1, \theta', \phi')| \leqq 1$, we find $U(1, \theta', \phi') = (-1)^k$ identically in ϕ'. Consequently, by use of the notation (8), (9), we have for the values of θ' mentioned before, $A_\nu(1, \theta') = B_\nu(1, \theta') = 0$, $\nu = 1, 2, \cdots, n$, or if $\sin\theta' \neq 0$,

$$\sum_{\mu=\nu}^{n} a_{\mu\nu} P_\mu^{(\nu)}(\cos\theta') = \sum_{\mu=\nu}^{n} b_{\mu\nu} P_\mu^{(\nu)}(\cos\theta') = 0.$$

These equations hold in particular for $\theta' = \pi/n, 2\pi/n, \cdots, (n-1)\pi/n$, so that $A_\nu(1, \theta)$ and $B_\nu(1, \theta)$ must vanish identically for ν greater than or equal to 2, and $A_1(1, \theta) = \text{const.} \sin n\theta$, $B_\nu(1, \theta) = \text{const.} \sin n\theta$. Finally, $U(1, \theta, \phi) = \cos n\theta + c \sin n\theta \cos(\phi - \phi_0)$, c and ϕ_0 constant, and as a consequence of $|U| \leqq 1$ we find $c = 0$.

CALCULATION OF ρ_n AND σ_n

1. The constants ρ_n and σ_n involved in the previous considerations, can be calculated in various ways. A comparatively simple procedure is based on the well known formulas

$$\begin{aligned} &\lim_{\epsilon\to+0} \{Q_n(u + i\epsilon) - Q_n(u - i\epsilon)\} = -i\pi P_n(u), \\ (23)\qquad &\lim_{\epsilon\to+0} \{Q_n'(u + i\epsilon) - Q_n'(u - i\epsilon)\} = -i\pi P_n'(u), \end{aligned}$$

where $-1 < u < +1$ and $Q_n(\xi)$ denotes Legendre's function of the second kind. Let $z = re^{i\theta}$, $0 < r < 1$, $0 < \theta < \pi$; then (23) can be written as follows:

$$\begin{aligned} &P_n(\cos\theta) = \frac{2}{\pi} \lim_{r\to1-0} \Im Q_n(\tfrac{1}{2}(z + z^{-1})), \\ (24)\qquad &P_n'(\cos\theta) = \frac{2}{\pi} \lim_{r\to1-0} \Im Q_n'(\tfrac{1}{2}(z + z^{-1})), \end{aligned}$$

so that

$$
\begin{aligned}
&\frac{1}{4}\pi\sin\theta\sum_{\mu=1}^{n}\mu(2\mu+1)P_\mu(\cos\theta)\\
&\qquad=\lim_{r\to 1-0}\frac{\pi}{4}\,\frac{z-z^{-1}}{2i}\,\frac{2}{\pi}\,\Im\sum_{\mu=1}^{n}\mu(2\mu+1)Q_\mu(\tfrac{1}{2}(z+z^{-1}))\\
&\qquad=\lim_{r\to 1-0}\Re\left\{\frac{1-z^2}{4z}\sum_{\mu=1}^{n}\mu(2\mu+1)Q_\mu(\tfrac{1}{2}(z+z^{-1}))\right\},\\
(25)\quad&\frac{1}{8}\pi\sin^2\theta\sum_{\mu=0}^{n}(2\mu+1)P_\mu'(\cos\theta)\\
&\qquad=\lim_{r\to 1-0}\frac{\pi}{8}\left(\frac{z-z^{-1}}{2i}\right)^2\frac{2}{\pi}\,\Im\sum_{\mu=0}^{n}(2\mu+1)Q_\mu'(\tfrac{1}{2}(z+z^{-1}))\\
&\qquad=-\lim_{r\to 1-0}\Im\left\{\frac{(1-z^2)^2}{16z^2}\sum_{\mu=0}^{n}(2\mu+1)Q_\mu'(\tfrac{1}{2}(z+z^{-1}))\right\}.
\end{aligned}
$$

Now $Q_\mu((z+z^{-1})/2)$ is a power series of z starting with $z^{\mu+1}$; therefore $z^{-2}Q_\mu'((z+z^{-1})/2)$ is a power series starting with z^μ. Adding to the sums on the right-hand side of (25) the corresponding terms with $\mu>n$, we do not influence the coefficients of $z, z^2, \cdots, z^n$. From this, we conclude that the n first coefficients of the expansions (12) agree with the corresponding coefficients of

$$
\begin{aligned}
&f(z)=\frac{1-z^2}{4z}\sum_{\mu=1}^{\infty}\mu(2\mu+1)Q_\mu(\tfrac{1}{2}(z+z^{-1})),\\
(26)\quad&g(z)=-\frac{(1-z^2)^2}{16z^2}\sum_{\mu=0}^{\infty}(2\mu+1)Q_\mu'(\tfrac{1}{2}(z+z^{-1})).
\end{aligned}
$$

In particular, ρ_n and σ_n are the coefficients of z^n in the power series expansions of the functions (26), regular for $|z|<1$.

2. In order to compute the functions (26), we start from the classical formula

$$
\begin{aligned}
\frac{1}{\xi-1}&=\sum_{\mu=0}^{n}(2\mu+1)Q_\mu(\xi)+\sum_{\mu=n+1}^{\infty}(2\mu+1)Q_\mu(\xi)\\
(27)\quad&=\sum_{\mu=0}^{n}(2\mu+1)Q_\mu(\xi)+\frac{n+1}{\xi-1}\{Q_n(\xi)-Q_{n+1}(\xi)\}\\
&=\sum_{\mu=0}^{n}(2\mu+1)Q_\mu(\xi)+r_{n+1}(\xi).
\end{aligned}
$$

Here $\xi=(z+z^{-1})/2$, $|z|<1$, is in the complex plane cut along the segment $[-1,+1]$. Now

$$\sum_{\mu=1}^{\infty} \mu(2\mu + 1)Q_\mu(\xi) = \sum_{\mu=1}^{\infty} \mu(r_\mu(\xi) - r_{\mu+1}(\xi)) = r_1(\xi) + r_2(\xi) + r_3(\xi) + \cdots$$
$$= \frac{1}{\xi - 1} \sum_{\mu=1}^{\infty} \mu(Q_{\mu-1}(\xi) - Q_\mu(\xi)) = \frac{1}{\xi - 1} \sum_{\mu=0}^{\infty} Q_\mu(\xi),$$

so that from the well known integral representation of $Q_\mu(\xi)$, by use of the generating function of $P_\mu(t)$,

$$\sum_{\mu=1}^{\infty} \mu(2\mu + 1)Q_\mu(\xi) = \frac{1}{\xi - 1} \sum_{\mu=0}^{\infty} \frac{1}{2} \int_{-1}^{+1} \frac{P_\mu(t)}{\xi - t} dt = \frac{2^{-3/2}}{\xi - 1} \int_{-1}^{+1} \frac{(1 - t)^{-1/2}}{\xi - t} dt$$
$$= \frac{2^{-1/2}}{(\xi - 1)^{3/2}} \operatorname{arc} \tan \left(\frac{2}{\xi - 1}\right)^{1/2}$$
$$= \frac{2z^{3/2}}{(1 - z)^3} \operatorname{arc} \tan \left(\frac{2z^{1/2}}{1 - z}\right)$$
$$= \frac{4z^{3/2}}{(1 - z)^3} \operatorname{arc} \tan (z^{1/2}).$$

Consequently,

$$f(z) = \frac{1 + z}{(1 - z)^2} z^{1/2} \operatorname{arc} \tan (z^{1/2}) = \frac{1 + z}{(1 - z)^2} \frac{z^{1/2}}{2i} \log \frac{1 + iz^{1/2}}{1 - iz^{1/2}}$$
$$= \sum_{\nu=0}^{\infty} (2\nu + 1)z^\nu \cdot \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{2\nu - 1} z^\nu,$$

(28)

$$\rho_n = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{2\nu - 1} (2n - 2\nu + 1)$$
$$= 2n\left(\frac{1}{1} - \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{2n - 1}\right) + \begin{cases} 0, & n \text{ even}, \\ -1, & n \text{ odd}, \end{cases}$$

which agrees with the expression in the right-hand member of the inequality (3).

3. A simpler argument furnishes from (27)

(29)

$$\sum_{\mu=0}^{\infty} (2\mu + 1)Q_\mu'(\xi) = -(\xi - 1)^{-2} = -4z^2(1 - z)^{-4},$$
$$g(z) = \frac{1}{4}\left(\frac{1 + z}{1 - z}\right)^2, \quad \sigma_n = n.$$

4. It can be easily seen that

(30) $$\rho_n > 2^{1/2}\sigma_n \qquad \text{for } n \geqq 4.$$

Indeed, we write

$$\frac{\rho_n}{2n} = 1 - \frac{1}{3} + \cdots + \frac{(-1)^{n-1}}{2n-1} + \frac{\epsilon_n}{2n},$$

where $\epsilon_n = 0$ or -1, according as n is even or odd. If n is even, we find, $n \geqq 4$,

$$\frac{\rho_n}{2n} \geqq 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \frac{76}{105} > 2^{-1/2}.$$

If n is odd, $n \geqq 5$,

$$\frac{\rho_n}{2n} = \frac{\rho_{n-1}}{2(n-1)} + \frac{1}{2n-1} - \frac{1}{2n} > \frac{\rho_{n-1}}{2(n-1)} > 2^{-1/2}.$$

Apparently, (30) does not hold for $n=1, 2, 3$. Indeed $\rho_1 = 1$, $\rho_2 = 8/3$, $\rho_3 = 21/5$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\sigma_3 = 3$.

Positivity of $M_n(\theta)$

Let $n \geqq 2$. The remaining part of the proof is devoted to the discussion of the trigonometric polynomial (4), that is, of

$$M_n(\theta) = 2\sum_{\nu=0}^{n-1}{}^{*}\left(\frac{\rho_{n-\nu}}{\rho_n}\cos n\theta\cos(n-\nu)\theta + \frac{n-\nu}{n}\sin n\theta\sin(n-\nu)\theta\right),$$

where the symbol $\sum^*$ indicates that the term $\nu = 0$ must be multiplied by $1/2$. We want to show that $M_n(\theta) > 0$.

1. Let $0 < \theta < \pi$. For the function $f(z)$ in (28), we have $f(0) = 0$, so that

$$\begin{aligned} 2\sum_{\nu=0}^{n-1}{}^{*}\rho_{n-\nu}\cos(n-\nu)\theta &= \frac{1}{2\pi i}\int\frac{f(z)}{z^{n+1}}\left\{2\sum_{\nu=0}^{n-1}{}^{*}z^{\nu}\cos(n-\nu)\theta\right\}dz \\ &= \frac{1}{2\pi i}\int\frac{f(z)}{z^{n+1}}\left\{\cos n\theta + 2\sum_{\nu=1}^{\infty}z^{\nu}\cos(n-\nu)\theta\right\}dz \qquad (31)\\ &= \frac{1}{2\pi i}\int\frac{f(z)}{z^{n+1}}\,\frac{(1-z^2)\cos n\theta + 2z\sin n\theta\sin\theta}{1-2z\cos\theta+z^2}\,dz. \end{aligned}$$

In an arbitrary domain not containing $z=-1$, the function $f(z)$ has the only singularity $z=+1$; an additional singularity of the last integrand is at $z=e^{\pm i\theta}$. The integrand being single-valued around $z=+1$, $e^{\pm i\theta}$, we can choose as a contour the following: A "large" arc $|z| = R$, $|\arg z| \leqq \pi - \epsilon$, and (approximately) the segment $-R$, -1 of the negative real axis, counting twice and encircling $z=-1$ in the negative sense. The residues at $z=+1$, $e^{\pm i\theta}$ have to be taken into consideration.

Now, since $f(z)=O(|z|^{-1/2})$ as $z\to\infty$ (compare the second form of $f(z)$ given in (28)), the contribution of the large arc can be disregarded. The residues furnish

$$(32)\qquad \begin{aligned} &-\frac{d}{dz}\left\{(1+z)z^{-n-1/2}\arctan(z^{1/2})\frac{(1-z^2)\cos n\theta+2z\sin n\theta\sin\theta}{1-2z\cos\theta+z^2}\right\}_{z=+1}\\ &-2\Re\left\{e^{-i(n+1)\theta}f(e^{i\theta})\frac{(1-e^{2i\theta})\cos n\theta+2e^{i\theta}\sin n\theta\sin\theta}{e^{i\theta}-e^{-i\theta}}\right\}. \end{aligned}$$

The first term is

$$\begin{aligned} (-\pi/4+(2n+1)\pi/4-1/2)&\frac{2\sin n\theta\sin\theta}{2(1-\cos\theta)}-\frac{\pi}{2}\frac{-2\cos n\theta}{2(1-\cos\theta)}\\ &=\frac{(n\pi-1)\sin n\theta\sin\theta+\pi\cos n\theta}{4\sin^2(\theta/2)}. \end{aligned}$$

For the calculation of the second term, we note that for $0<\theta<\pi$

$$(33)\qquad \begin{aligned} f(e^{i\theta})&=\frac{1+e^{i\theta}}{(1-e^{i\theta})^2}\frac{e^{i\theta/2}}{2i}\{\log\cot[(\theta+\pi)/4]+i\pi/2\}\\ &=\frac{\cos(\theta/2)}{4\sin^2(\theta/2)}\{i\log\cot[(\theta+\pi)/4]-\pi/2\}, \end{aligned}$$

so that the second term of (32) is

$$-2\Re\{e^{-i(n+1)\theta}f(e^{i\theta})(-e^{i(n+1)\theta})\}=2\Re f(e^{i\theta})=\frac{-\pi\cos(\theta/2)}{4\sin^2(\theta/2)}.$$

Finally, we consider

$$(34)\qquad \frac{1}{2\pi i}\int_{-\infty}^{(-1,-)}\frac{f(z)}{z^{n+1}}\frac{(1-z^2)\cos n\theta+2z\sin n\theta\sin\theta}{1-2z\cos\theta+z^2}\,dz.$$

On the "upper" border of the line $-\infty,\ -1$ we have, $z=-t$, $1<t<+\infty$,

$$f(z)=\frac{1-t}{(1+t)^2}\frac{t^{1/2}}{2}\left\{\log\left(\frac{t^{1/2}-1}{t^{1/2}+1}\right)+i\pi\right\},$$

whereas on the "lower" border, $i\pi$ must be replaced by $-i\pi$. Consequently, (34) will be

$$\frac{(-1)^n}{2}\int_1^\infty\frac{t-1}{(t+1)^2}t^{-n-1/2}\frac{(1-t^2)\cos n\theta-2t\sin n\theta\sin\theta}{1+2t\cos\theta+t^2}\,dt,$$

so that

$$
(35)\quad \begin{aligned} 2\sum_{\nu=0}^{n-1}{}^{*}\rho_{n-\nu}\cos(n-\nu)\theta &= \frac{(n\pi-1)\sin n\theta\sin\theta+\pi\cos n\theta-\pi\cos(\theta/2)}{4\sin^2(\theta/2)} \\ &\quad + \frac{(-1)^n}{2}\int_1^\infty \frac{t-1}{(t+1)^2}\,t^{-n-1/2}\,\frac{(1-t^2)\cos n\theta-2t\sin n\theta\sin\theta}{1+2t\cos\theta+t^2}\,dt. \end{aligned}
$$

2. According to a well known formula,

$$
\begin{aligned} 1+2\sum_{\nu=1}^{n-1}\cos\nu\theta+\cos n\theta &= \frac{\sin(2n-1)(\theta/2)}{2\sin(\theta/2)}+\frac{\sin(2n+1)(\theta/2)}{2\sin(\theta/2)} \\ &= \sin n\theta\cot(\theta/2), \end{aligned}
$$

so that

$$
\begin{aligned} 2\sum_{\nu=0}^{n-1}{}^{*}(n-\nu)\sin(n-\nu)\theta &= 2\sum_{\nu=1}^{n-1}\nu\sin\nu\theta+n\sin n\theta \\ &= -n\cos n\theta\cot(\theta/2)+\frac{\sin n\theta}{2\sin^2(\theta/2)}\cdot \end{aligned}
$$

This leads, in view of (35), to the following important representation:

$$
(36)\quad \begin{aligned} \rho_n M_n(\theta) &= \frac{(n\pi-1)\sin n\theta\sin\theta+\pi\cos n\theta-\pi\cos(\theta/2)}{4\sin^2(\theta/2)}\cos n\theta \\ &\quad + \frac{(-1)^{n-1}}{2}\cos n\theta\int_1^\infty\frac{t-1}{(t+1)^2}\,t^{-n-1/2}\,\Re\left\{e^{in\theta}\frac{1-t^{-1}e^{i\theta}}{1+t^{-1}e^{i\theta}}\right\}dt \\ &\quad - \tfrac{1}{2}\rho_n\sin 2n\theta\cot(\theta/2)+\frac{\rho_n}{2n}\,\frac{\sin^2 n\theta}{\sin^2(\theta/2)}\cdot \end{aligned}
$$

3. Now, we first establish the inequality

$$
(37)\quad \left|\frac{(-1)^{n-1}}{2}\cos n\theta\int_1^\infty\frac{t-1}{(t+1)^2}\,t^{-n-1/2}\,\Re\left\{e^{in\theta}\frac{1-t^{-1}e^{i\theta}}{1+t^{-1}e^{i\theta}}\right\}dt\right| < \frac{1}{4n}\cdot
$$

Indeed,

$$
\left|\Re\left\{e^{in\theta}\frac{1-t^{-1}e^{i\theta}}{1+t^{-1}e^{i\theta}}\right\}\right| \leqq \frac{1+t^{-1}}{1-t^{-1}},
$$

and

$$
\frac{1}{2}\int_1^\infty\frac{t^{-n-1/2}}{t+1}\,dt < \frac{1}{4}\int_1^\infty t^{-n-1}dt = \frac{1}{4n}\cdot
$$

4. Next, we proceed to a discussion of the quantities ρ_n. If s_n denotes the

nth partial sum of the series $1/1-1/3+1/5-\cdots$, and $\epsilon_n=0$ or -1 according as n is even or odd, we have

$$\begin{aligned}\rho_n &= 2ns_n+\epsilon_n = 2n\left(\pi/4-\sum_{\nu=n+1}^{\infty}\frac{(-1)^{\nu-1}}{2\nu-1}\right)+\epsilon_n\\ &= n\pi/2+\epsilon_n+(-1)^{n+1}2n\left\{\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\right.\\ &\qquad\left.+\left(\frac{1}{2n+5}-\frac{1}{2n+7}\right)+\cdots\right\}.\end{aligned}$$

We compare the last series with

$$\frac{1}{2}\left(\frac{1}{2n}-\frac{1}{2n+4}\right)+\frac{1}{2}\left(\frac{1}{2n+4}-\frac{1}{2n+8}\right)+\cdots=\frac{1}{4n}\cdot$$

Since

$$\begin{aligned}0&<\frac{1}{2}\left(\frac{1}{2n}-\frac{1}{2n+4}\right)-\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)\\ &=\frac{6}{2n(2n+1)(2n+3)(2n+4)}<\frac{3}{8n^4},\end{aligned}$$

we find

$$\rho_n = n\pi/2+\epsilon_n+(-1)^{n+1}/2+\delta_n = \tfrac{1}{2}(n\pi-1)+\delta_n, \tag{38}$$

where

$$\begin{aligned}|\delta_n| &< \frac{3n}{4}\left(\frac{1}{n^4}+\frac{1}{(n+2)^4}+\frac{1}{(n+4)^4}+\cdots\right)\\ &< \frac{3}{4}\frac{1}{n^3}+\frac{3}{8}n\int_n^{\infty}\frac{dx}{x^4}=\frac{3}{4}\frac{1}{n^3}+\frac{1}{8}\frac{1}{n^2}\cdot\end{aligned}\tag{39}$$

Incidentally, sgn $\delta_n=(-1)^n$.

5. Returning again to (36), we obtain

$$\begin{aligned}\rho_nM_n(\theta) &= \frac{(n\pi-1)\sin n\theta\sin\theta+\pi\cos n\theta-\pi\cos(\theta/2)}{4\sin^2(\theta/2)}\cos n\theta\\ &\quad-\tfrac{1}{4}(n\pi-1)\sin 2n\theta\cot(\theta/2)+\frac{n\pi-1}{4n}\frac{\sin^2 n\theta}{\sin^2(\theta/2)}+\delta_n'\\ &= \frac{\pi(1-\cos(\theta/2)\cos n\theta)-n^{-1}\sin^2 n\theta}{4\sin^2(\theta/2)}+\delta_n',\end{aligned}\tag{40}$$

where

$$
(41)\qquad
\begin{aligned}
|\delta_n'| &< \frac{1}{4n} + |\delta_n| \left(\frac{1}{2} |\sin 2n\theta| \cot(\theta/2) + \frac{1}{2n} \frac{\sin^2 n\theta}{\sin^2(\theta/2)} \right) \\
&< \frac{1}{4n} + |\delta_n| \left(\frac{1}{2} \cdot 2n\theta \cdot \frac{2}{\theta} + \frac{1}{2n} \cdot 4n^2 \right) = \frac{1}{4n} + 4n|\delta_n|,
\end{aligned}
$$

or, according to (39),

$$
(42)\qquad |\delta_n'| < 3/(4n) + 3/n^2.
$$

6. In order to prove the positivity of (40), we distinguish two cases.

(a) Let $(1/2)n\pi \cos(\theta/2) < 1$. Then

$$
(43)\qquad \pi(1 - \lambda \cos(\theta/2)) - n^{-1}(1 - \lambda^2),
$$

where λ varies between -1 and $+1$, has a minimum at $\lambda = (1/2)n\pi \cos(\theta/2)$ which is $\pi - n^{-1} - (1/4)n\pi^2 \cos^2(\theta/2)$. Therefore, we only have to show that

$$
\frac{\pi - n^{-1} - (1/4)n\pi^2 \cos^2(\theta/2)}{4 \sin^2(\theta/2)} > \frac{3}{4n} + \frac{3}{n^2}.
$$

Since $\pi - n^{-1} - n\pi^2/4 < 0$, the expression on the left-hand side is a decreasing function of $\cos(\theta/2)$, so that its minimum is attained for $\cos(\theta/2) = 2/n\pi$. But

$$
\frac{\pi - n^{-1} - n^{-1}}{4(1 - 4/n^2\pi^2)} = \frac{1}{4}\pi\left(1 + \frac{2}{n\pi}\right)^{-1} > \frac{3}{4n} + \frac{3}{n^2};
$$

this inequality is true for $n \geqq 3$.

(b) Let $(1/2)n\pi \cos(\theta/2) > 1$. Then (43) is decreasing and its minimum is reached for $\lambda = +1$, which is $\pi(1 - \cos(\theta/2))$. Therefore, it is sufficient to show that

$$
\frac{\pi(1 - \cos(\theta/2))}{4 \sin^2(\theta/2)} = \frac{\pi}{4(1 + \cos(\theta/2))} \geqq \frac{\pi}{8} > \frac{3}{4n} + \frac{3}{n^2},
$$

which is true for $n \geqq 4$.

It remains to show that for $n = 2$ and $n = 3$, we have $\pi/8 > |\delta_n'|$, or using (41), $\pi/8 > 1/(4n) + 4n|\delta_n|$. Since $\rho_2 = 8/3$, $\rho_3 = 21/5$, we find from (38) $0 < \delta_2 = 19/6 - \pi$, $0 < -\delta_3 = 3\pi/2 - 47/10$, so that the statements are $\pi > 1 + 64(19/6 - \pi)$, $\pi > 2/3 + 96(3\pi/2 - 47/10)$. Both inequalities can easily be verified.

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